Lecture 11: CW complex

Recall that $S^{n-1} \hookrightarrow D^n$ is a cofibration satisfying HEP, where, D^n is the n-disk and $S^{n-1} = \partial D^n$ is its boundary, the $(n-1)$-sphere. Let

$$e^n = (D^n)^\circ = D^n - \partial D^n$$

denotes the interior of D^n, the open disk known as the n-cell.

The category of CW-complex consists of topological spaces that can be built from n-cells and behaves nicely just like $S^{n-1} \hookrightarrow D^n$. It is also large enough to cover most interesting examples.

Definition

A cell decomposition of a space X is a family

$$\mathcal{E} = \{e^n_\alpha | \alpha \in J_n\}$$

of subspaces of X such that each e^n_α is a n-cell and we have a disjoint union of sets

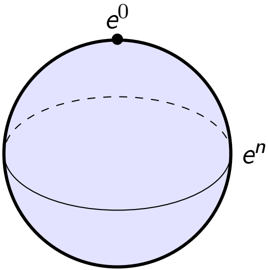
$$X = \coprod e^n_\alpha.$$

The n-skeleton of X is the subspace

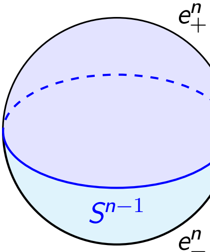
$$X^n = \coprod_{\alpha \in J_m, m \leq n} e^m_\alpha.$$

Example

Two cellular structures on S^n



$$S^n = e^0 \cup e^n$$



$$
\begin{aligned}
S^n &= e_+^n \cup e_-^n \cup S^{n-1} \\
&= \left(e_+^n \cup e_-^n\right) \cup \left(e_+^{n-1} \cup e_-^{n-1}\right) \cup \cdots \cup \left(e_+^0 \cup e_-^0\right)
\end{aligned}
$$

Definition

A CW complex is a pair $(X, \mathcal{E})$ of a Hausdorff space X with a cell decomposition such that

1. Characteristic map: for each n-cell e^n_α, there is a map
$$\Phi_{e^n_\alpha} : D^n \to X$$
such that the restriction of $\Phi_{e^n_\alpha}$ to $(D^n)^\circ$ is a homeomorphism to e^n_α and $\Phi_{e^n_\alpha}(S^{n-1}) \subset X^{n-1}$.
2. C=Closure finiteness: for any cell $e \in \mathcal{E}$ the closure $\bar{e}$ intersects only a finite number of cells in $\mathcal{E}$.
3. W=Weak topology: a subset $A \subset X$ is closed if and only if $A \cap \bar{e}$ is closed in $\bar{e}$ for each $e \in \mathcal{E}$.

We say X is n-dim CW complex if the maximal dimension of cells in $\mathcal{E}$ is n (n could be ∞).

Note that the Hausdorff property of X implies that

$$\bar{e} = \Phi_e(D^n) \quad \text{for each cell } e \in \mathcal{E}.$$

The surjective map $\Phi_e : D^n \to \bar{e}$ is a quotient since D^n is compact and $\bar{e}$ is Hausdorff. Let us denote the full characteristic maps

$$\Phi : \coprod_{e \in \mathcal{E}} D^n \xrightarrow{\coprod \Phi_e} X.$$

Then the weak topology implies that Φ is a quotient map.

Proposition

Let $(X, \mathcal{E})$ be a CW complex. Then $f\colon X \to Y$ is continuous if and only if

$$f \circ \Phi_e : D^n \to Y$$

is continuous for each $e \in \mathcal{E}$.

Proposition

Let $(X, \mathcal{E})$ be a CW complex. Then any compact subspace of X meets only finitely many cells.

Proof.

Assume K is a compact subspace of X which meets infinitely many cells. Let $x_i \in K \cap e_i$, $i = 1, 2, \cdots$, where e_i's are different cells. Let

$$Z_m = \{x_m, x_{m+1}, \cdots\}, \quad m \geq 1.$$

By the closure finiteness, Z_m intersects each closure $\bar{e}$ by finite points, hence closed in $\bar{e}$ by the Hausdorff property. By the weak topology, Z_m is a closed subset of X, hence closed in K. Observe

$$\bigcap_{m \geq 1} Z_m = \emptyset$$

but any finite intersection of Z_m's is non-empty. This contradicts the compactness of K. □

Proposition

Let $(X, \mathcal{E})$ be a CW complex and X^n be the n-skeleton. Then X is the colimit (i.e. direct limit) of the telescope diagram

$$X^1 \to X^2 \to \cdots \to X^n \to \cdots$$

Proof.

This is because $f\colon X \to Y$ is continuous if and only if $f\colon X^n \to Y$ is continuous for each n.

$\square$

Proposition

Let $(X, \mathcal{E})$ be a CW complex. Then X is compactly generated weak Hausdorff.

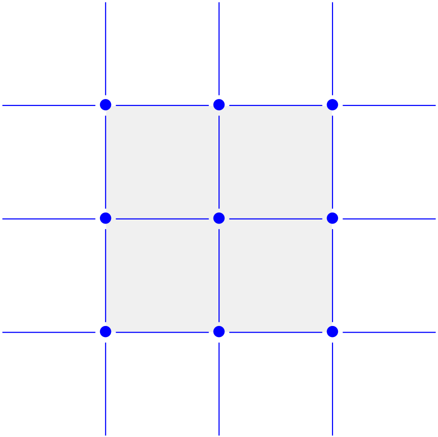
Proof.

X is Hausdorff, hence also weak Hausdorff.

We check X is compactly generated. Assume $Z \subset X$ is k-closed. Since the closure of each cell $\bar{e}$ is compact Hausdorff, $Z \cap \bar{e}$ is closed in $\bar{e}$. The weak topology implies that Z is closed in X. $\square$

Example

Gride/cube decomposition of $\mathbb{R}^n$ into n-cube $I^n \simeq D^n$.



Example

$\mathbb{CP}^n$: $(\mathbb{C}^{n+1} - \{0\})/\sim$ and we have

$$\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \cdots \mathbb{CP}^{n-1} \subset \mathbb{CP}^n \subset \cdots \subset \mathbb{CP}^\infty.$$

Moreover,

$$\begin{aligned}\mathbb{CP}^n - \mathbb{CP}^{n-1} &= \{[z_0, \ldots, z_n] \mid z_n \neq 0\} \\ &\simeq \mathbb{C}^n \simeq e^{2n}.\end{aligned}$$

Thus $\mathbb{CP}^n$ has one cell in every even dimension from 0 to $2n$ with characteristic map

$$\begin{aligned}\Phi_{2n}\colon D^{2n} &\longrightarrow \mathbb{CP}^n \\ (z_0, \ldots, z_n) &\mapsto \left[z_0, \ldots, z_{n-1}, \sqrt{1 - \Sigma_{i=0}^{n-1}|z_i|^2}\right]\end{aligned}$$

Definition

A subcomplex $(X', \mathcal{E}')$ of the CW complex $(X, \mathcal{E})$ is a closed subspace $X' \subset X$ with a cell decomposition $\mathcal{E}' \subset \mathcal{E}$. We will just write $X' \subset X$ when the cell decomposition is clear. We will also write $X' = |\mathcal{E}'|$.

Equivalently, a subcomplex is given by a subset $\mathcal{E}' \subset \mathcal{E}$ such that

$$e_1 \in \mathcal{E}', e_2 \in \mathcal{E}, \bar{e}_1 \cap e_2 \neq \emptyset \Longrightarrow e_2 \in \mathcal{E}'.$$

Definition

Given $f\colon S^{n-1} \to X$. Consider the push-out

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, hook] \\ D^n \arrow[r, "\Phi_f"] & D^n \coprod_f X \end{tikzcd}$$

We say $D^n \coprod_f X$ is obtained by attaching an n-cell to X.

Φ_f is called the characteristic map of the attached n-cell.

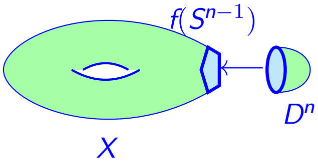


图: Attaching a cell

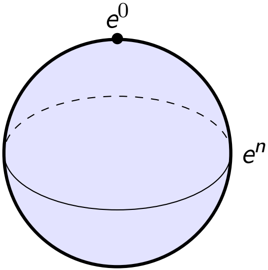
More generally, if we have a set of maps $f_\alpha : S^{n-1} \to X$, then the push-out

$$\begin{tikzcd}
\coprod_\alpha S^{n-1} \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, hook] \\
\coprod_\alpha D^n \arrow[r, "\Phi_f"] & (\coprod D^n) \coprod_f X
\end{tikzcd} \qquad f = \coprod f_\alpha$$

is called attaching n-cells to X.

Example

The n-sphere S^n can be obtained by attaching a n-cell to a point.



Proposition

Let $(X, \mathcal{E})$ be a CW complex, and $\mathcal{E} = \coprod \mathcal{E}^n$ where $\mathcal{E}^n$ is the set of n-cells. Then the diagram

$$\begin{tikzcd} \coprod\limits_{e \in \mathcal{E}^n} S^{n-1} \arrow[r, "\partial \Phi^n"] \arrow[d, hook] & X^{n-1} \arrow[d] \\ \coprod\limits_{e \in \mathcal{E}^n} D^n \arrow[r, "\Phi^n"] & X^n \end{tikzcd} \qquad \Phi^n = \coprod_{e \in \mathcal{E}^n} \Phi_e$$

is a push-out. In particular, X^n is obtained from X^{n-1} by attaching n-cells in X.

Proof.

This follows from the fact that X^{n-1} is a closed subspace of X^n and the weak topology. $\square$

The converse is also true. The next theorem can be viewed as an alternate definition of CW complex.

Theorem

Suppose we have a sequence of spaces

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^n \subset X^{n+1} \subset \cdots$$

where X^n is obtained from X^{n-1} by attaching n-cells. Let

$$X = \cup_{n \geq 0} X^n$$

be the union with the weak topology: $A \subset X$ is closed if and only if $A \cap X^n$ is closed in X^n for each n. Then X is a CW complex.

The theorem follows directly from the next lemma.

Lemma

Let X be a $(n-1)$-dim CW complex and Y is obtained from X by attaching n-cells. Then Y is a n-dim CY complex.

Proof:

C: Closure finiteness follows from the fact that S^{n-1} is compact.

W: Weak topology follows from the push-out construction.

We need to check the Hausdorff property of Y.

H: The Hausdroff property of Y. Take $x, y \in Y$. If x lies in an n-cell, then it is easy to separate x from y. Otherwise, let $x, y \in X$ and take their open neighbourhoods U, V in X that separate them. Consder attaching the n-cells via the push-out:

$$\begin{tikzcd} \coprod_\alpha S^{n-1} \arrow[r, "\coprod_\alpha g_\alpha"] \arrow[d, hook] & X \arrow[d, hook] \\ \coprod_\alpha D^n \arrow[r, "\coprod_\alpha \Phi_\alpha"] & Y \end{tikzcd}$$

Then $g_\alpha^{-1}(U), g_\alpha^{-1}(V)$ are open in S^{n-1}. Take their open neighbourhoods U_α, V_α in D^n, i.e.

$$U_\alpha \cap S^{n-1} = g_\alpha^{-1}(U), \qquad V_\alpha \cap S^{n-1} = g_\alpha^{-1}(V)$$

such that $U_\alpha \cap V_\alpha = \emptyset$. Then $U \cup (\bigcup_\alpha U_\alpha)$ and $V \cup (\bigcup_\alpha V_\alpha)$ are separated neighbourhood of x, y.

Definition

Let A be a subspace of X. A CW decomposition of (X, A) consists of a sequence

$$A = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X$$

such that X^n is obtained from X^{n-1} by attaching n-cells and X carries the weak topology with respect to the subspaces X^n. The pair (X, A) is called a relative CW complex.

Note that for a relative CW complex (X, A), A itself may not be a CW complex.

Proposition

Let (X, A) be a relative CW complex. Then $A \subset X$ is a cofibration.

Proof.

$S^{n-1} \hookrightarrow D^n$ is a cofibration, and cofibration is preserved under push-out, so each

$$X^{n-1} \to X^n$$

is a cofibration. The proposition follows since composition of cofibrations is a cofibration. □

Product of CW complexes

Let $(X, \mathcal{E}), (Y, \tilde{\mathcal{E}})$ be two CW complexes. We can define a cellular structure on $X \times Y$ with n-skeleton

$$(X \times Y)^n = \{e^k_\alpha \times \tilde{e}^l_\beta | 0 \leq k + l \leq n, \quad e^k_\alpha \in \mathcal{E}, \tilde{e}^l_\beta \in \tilde{\mathcal{E}}\}$$

and characteristic maps

$$\Phi^{k,l}_{\alpha,\beta} = (\Phi^l_\alpha, \Phi^l_\beta) : D^{k+l}_{\alpha,\beta} \to X \times Y.$$

Here we use the fact that $D^{k+l}_{\alpha,\beta} \equiv D^k_\alpha \times D^l_\beta$ topologically.

Example

Cellular decomposition for $S^1 \times S^1$.

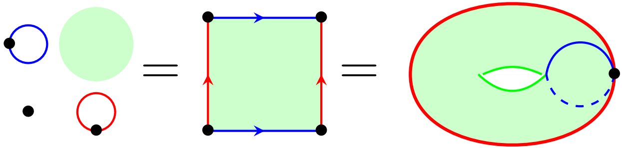


图: Cellular decomposition for $S^1 \times S^1$

This natural cellular structure is closure finite. However, the product topology on $X \times Y$ may not be the same as the weak topology, so the topological product may not be a CW complex.

Observe that X, Y are compactly generated weak Hausdorff, and we can take their categorical product in the category $\underline{\mathscr{T}}$. Then this compactly generated product will have the weak topology, and becomes a CW complex.

Proposition

Assume X is compactly generated and Y is locally compact Hausdorff, then the categorical product of X and Y in $\mathscr{T}$ is the same as the categorical product in **Top** (i.e. the topological product).

As a consequence, we have

Theorem

Let $X.Y$ be CW complexes and Y be locally compact. Then the topological product $X \times Y$ is a CW complex.

Example

If X is a CW complex, then $X \times I$ is a CW complex.

Definition

A CW complex X is called locally finite if each point in X has an open neighborhood that intersects only finite many cells.

Locally finite CW complexes are locally compact Hausdorff.

Corollary

Let $X.Y$ be CW complexes and Y be locally finite. Then the topological product $X \times Y$ is a CW complex.